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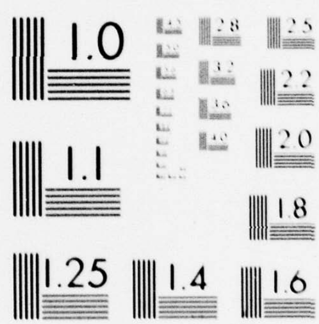

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THE COMPUTATION OF TIME DIFFERENCE ERROR PROBABILITIES

CENTER FOR NAVAL ANALYSES
1401 Wilson Boulevard
Arlington, Virginia 22209
Operations Evaluation Group

By: Peter J. Butterly

January 1979

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In many naval systems position is estimated from multiple time differences. System accuracy is thus dependent on the errors in these quantities. Time differences, however, are not quantities for which adequate error information is generally available or readily ascertained experimentally. Instead, it is derived from a knowledge of the errors in the component times from which the differences are formed.

In this research contribution, a formal treatment of the time → time difference transformation is developed for the case where differences are formed with respect Over

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20 to a single time. The effect of changing this reference is investigated and expressions for computing error probabilities for multiple time differences are derived under conditions related to current applications.

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INTRODUCTION

Many naval systems use time difference methods to estimate position. While the arrival time of a transmitted signal depends on the transmission time and the length of the transmission path, differences of arrival times do not depend on the unknown time at which the transmission takes place. Consequently, estimates of the position of a transmitter or receiver are based on observed time differences, and the quality of the estimate depends on the errors in these quantities, that is, on the joint time difference error density.

Although a knowledge of this density function is required for a quantitative treatment of any aspect of the estimation problem, it will seldom be directly available. When the system is calibrated, measurements performed on transmissions between known locations or over known paths provide data on the errors in the observed arrival times. This process yields a joint density for the errors in the arrival times, and the corresponding density for the differences in the arrival times must be derived from it. A formulation of the density transformation and some examples of its application are given in this research contribution.

FORMULATION OF THE TRANSFORMATION

If there are n observed arrival times there are $n(n-1)$ time differences. Of these, only $(n-1)$ may be chosen independently. It will therefore be assumed that one time is selected as a reference and that the remaining times, measured with respect to this reference, constitute the $(n-1)$ differences of interest.

Since taking differences is a linear operation which introduces correlation, matrix notation is convenient, particularly if related computer programs are written in a language suitable for processing numerical arrays. We therefore let

$$\underline{T} = \begin{bmatrix} T_1 & T_2 & \cdots & T_k & \cdots & T_n \end{bmatrix}'$$

and

$$\underline{t} = \begin{bmatrix} t_1 & t_2 & \cdots & t_k & \cdots & t_n \end{bmatrix}'$$

denote the observed and true values of the arrival times, respectively. The observed and true values of the differences in arrival times, where the k^{th} time is selected as reference, are then given by $\underline{\Delta}_k\underline{T}$ and $\underline{\Delta}_k\underline{t}$, respectively, where $\underline{\Delta}_k$ is the $(n-1) \times n$ matrix derived from the $n \times n$ identity matrix by replacing each zero element of the k^{th} column by -1 and deleting the k^{th} row. Thus, for example, if there are five observations and the fourth time is selected as reference, the observed time differences are given by

$$\begin{bmatrix} T_1 - T_4 \\ T_2 - T_4 \\ T_3 - T_4 \\ T_5 - T_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \underline{\Delta}_4\underline{T} .$$

Representing arrival time errors by

$$\underline{\varepsilon} = \left[\varepsilon_1 \;\; \varepsilon_2 \; \ldots \varepsilon_k \; \ldots \; \varepsilon_n\right]'$$

$$= \underline{T} - \underline{t} \; ,$$

and the time difference errors with the k^{th} time as reference by

$$\underline{e}_k = \left[e_{1k} \;\; e_{2k} \; \cdots \; e_{(k-1)k} \;\; e_{(k+1)k} \; \cdots \; e_{nk}\right]'$$

for k = 1, 2,..., n, we have

$$\begin{aligned} \underline{e}_k &= \underline{\Delta}_k \underline{T} - \underline{\Delta}_k \underline{t} \\ &= \underline{\Delta}_k \underline{\varepsilon} \; . \end{aligned} \tag{1}$$

This is the required error transformation. It permits the moments of the time difference error density to be computed from those of the arrival time error density. In particular,

$$E(\underline{e}_k) = \underline{\Delta}_k \underline{\mu} \tag{2}$$

where

$$\underline{\mu} = \left[\mu_1 \;\; \mu_2 \; \cdots \; \mu_k \; \cdots \; \mu_n\right]' \; ,$$

$$\mu_i = E(\varepsilon_i) \qquad i=1,\, 2,\, \ldots,\, k \,, \ldots,\, n$$

and

$$\underline{\Sigma}_{e_k} = \underline{\Delta}_k \, \underline{\Sigma}_\varepsilon \, \underline{\Delta}_k' \tag{3}$$

where $\underline{\Sigma}_\varepsilon$ and $\underline{\Sigma}_{e_k}$ are the covariance matrices for the error vectors $\underline{\varepsilon}$ and $\underline{e}_k$, respectively. If the elements of $\underline{\Sigma}_\varepsilon$ are given by

$$\operatorname{cov}(\varepsilon_i, \varepsilon_j) = \sigma_{ij} \qquad i,j = 1,2 \,, \ldots,\, n$$

the above expressions may be rewritten:

$$E(e_{ik}) = \mu_i - \mu_k \tag{4}$$

and

$$\mathrm{cov}(e_{ik}, e_{jk}) = \sigma_{ij} - \sigma_{ik} - \sigma_{kj} + \sigma_{kk} \tag{5}$$

$$i, j = 1, 2, \ldots, (k-1), (k+1), \ldots, n.$$

If arrival time errors are uncorrelated, then

$$\mathrm{cov}(e_{ik}, e_{jk}) = \sigma_i^2 \delta_{ij} + \sigma_k^2 . \tag{6}$$

The subsequent treatment of the case where the arrival time errors are jointly normal will make use of these results.

In formulating the joint error density transformation for the general case, we note that although the mapping $\underline{\varepsilon} \to \underline{e}_k$ is not one-to-one, the mapping $\underline{\varepsilon} \to \underline{e}_k^+$ is, where $\underline{e}_k^+$ is defined by

$$\underline{e}_k^+ = \left[e_{1k} \;\; e_{2k} \cdots e_{(k-1)k} \;\; \varepsilon_k \;\; e_{(k+1)k} \cdots e_{nk} \right]' .$$

The inverse mapping $\underline{e}_k^+ \to \underline{\varepsilon}$ has the form

$$\begin{aligned}
\varepsilon_1 &= e_{1k} + \varepsilon_k \\
\varepsilon_2 &= e_{2k} + \varepsilon_k \\
&\;\;\vdots \\
\varepsilon_k &= \varepsilon_k \\
&\;\;\vdots \\
\varepsilon_n &= e_{nk} + \varepsilon_k
\end{aligned}$$

and the Jacobian of the transformation is unity. Therefore, the required transformation is

$$P_{\underline{e}_k}(\underline{e}_k) = P_{\underline{e}_k}(e_{1k} \quad e_{2k} \cdots e_{(k-1)k} \quad e_{(k+1)k} \cdots e_{nk})$$

$$= \int P_{\underline{e}_k^+}(e_{1k} \quad e_{2k} \cdots e_{(k-1)k} \quad u \quad e_{(k+1)k} \cdots e_{nk})du$$

$$= \int P_{\underline{\varepsilon}}(e_{1k}+u \quad e_{2k}+u \cdots e_{(k-1)k}+u \quad u$$

$$e_{(k+1)k}+u \cdots e_{nk} + u)du \qquad (7)$$

where

$$P_{\underline{\varepsilon}}(\underline{\varepsilon}) = P_{\underline{\varepsilon}}(\varepsilon_1 \quad \varepsilon_2 \cdots \varepsilon_k \cdots \varepsilon_n)$$

is the joint error density for the arrival times.

If these errors are independent, then

$$P_{\underline{e}_k}(\underline{e}_k) = \int P_{\varepsilon_k}(u) \prod_{\substack{i=1 \\ i \neq k}}^{n} P_{\varepsilon_i}(e_{ik}+u)du \quad . \qquad (8)$$

These expressions provide the formal transformations for obtaining the joint time difference error density where the joint error density for the arrival times has a known mathematical form.

REFERENCE TIME SELECTION

In this section we consider the effect of changing the time selected as reference. For two such selections,

$$\underline{e}_k = \underline{\Delta}_k \, \underline{\varepsilon}$$

$$\underline{e}_r = \underline{\Delta}_r \, \underline{\varepsilon}$$

If $\underline{T}_{rk}$ has the property

$$\underline{T}_{rk} \, \underline{\Delta}_k = \underline{\Delta}_r \ ,$$

then

$$\underline{e}_r = \underline{\Delta}_r \, \underline{\varepsilon} = \underline{T}_{rk} \, \underline{\Delta}_k \, \underline{\varepsilon} = \underline{T}_{rk} \, \underline{e}_k \ .$$

That is, the linear operation denoted by $\underline{T}_{rk}$ provides the time differences with respect to the r^{th} time from those measured with respect to the k^{th} time. Since

$$\underline{\Delta}_k = \underline{T}_{kr} \, \underline{\Delta}_r = \underline{T}_{kr} \, \underline{T}_{rk} \, \underline{\Delta}_k \, ,$$

we have

$$\underline{T}_{kr} \, \underline{T}_{rk} = \underline{I}_{n-1}$$

where $\underline{I}_{n-1}$ is the (n-1)x(n-1) identity matrix, or,

$$T_{kr} = T_{rk}^{-1} \ .$$

While only this result will be used to investigate the effect of changing the reference time, we note that

$$\underline{\Delta}_k \, \underline{\Delta}_k' = \underline{\Delta}_r \, \underline{\Delta}_r' = \underline{I}_{n-1} + \underline{J}_{n-1}$$

where $\underline{I}_{n-1}$ is the (n-1)x(n-1) identify matrix and $\underline{J}_{n-1}$ is an (n-1)x(n-1) matrix with all elements equal to unity. Since it is readily shown that

$$(\underline{I}_{n-1} + \underline{J}_{n-1})^{-1} = \underline{I}_{n-1} - \frac{1}{n} \underline{J}_{n-1} ,$$

$$(\underline{\Delta}_k \underline{\Delta}_k')^{-1} = \underline{I}_{n-1} - \frac{1}{n} \underline{J}_{n-1}$$

exists for $n > 0$, $\underline{T}_{rk}$ may therefore be expressed in terms of difference operators alone by

$$\underline{T}_{rk} = \underline{\Delta}_r \underline{\Delta}_k^{-1} = \underline{\Delta}_r \underline{\Delta}_k' (\underline{\Delta}_k \underline{\Delta}_k')^{-1}$$

where[1]

$$\underline{\Delta}_k^{-1} = \underline{\Delta}_k' (\underline{\Delta}_k \underline{\Delta}_k')^{-1}$$

is the generalized inverse of the (n-1)x n matrix $\underline{\Delta}_k$.

To illustrate the effect of changing the reference time, let $\underline{\varepsilon}^*$ represent a sample value of the arrival time error. If the k^{th} time is selected as reference, the sample values for time difference errors are given by

$$\underline{e}_k^* = \underline{\Delta}_k \underline{\varepsilon}^* ,$$

and the joint error probability is given by $P_{\underline{e}_k}(\underline{e}_k^*)$. If, alternatively, the r^{th} time is selected as reference, a different time difference sample results, namely

$$\underline{e}_r^* = \underline{\Delta}_r \underline{\varepsilon}^* ,$$

and the joint error probability is then given by $P_{\underline{e}_r}(\underline{e}_r^*)$. But, since $\underline{e}_r = \underline{T}_{rk} \underline{e}_k$ and since the absolute value of the Jacobian of the transformation $\underline{T}_{rk}$ is unity,

$$P_{\underline{e}_r}(\underline{e}_r^*) = P_{\underline{e}_k}(\underline{e}_k^*) .$$

Joint time difference error probabilities are thus invariant under reference time selection.

---

[1] Graybill, F.A. "Introduction to Matrices With Applications in Statistics," Wadsworth Publishing Co., Inc., Belmont, Ca., 1969, p. 102.

NORMAL ERRORS

If the time of arrival errors are jointly normal, that is

$$P_{\underline{\varepsilon}}(\underline{\varepsilon}) \sim N(\underline{\mu}, \underline{\Sigma}_{\varepsilon}) ,$$

then, from equations 2 and 3, the time difference error density is given by

$$P_{\underline{e}_k}(\underline{e}_k) \sim N(\underline{\Delta}_k \, \underline{\mu}, \underline{\Delta}_k \, \underline{\Sigma}_{\varepsilon} \, \underline{\Delta}_k')$$

and the elements of the mean vector and covariance matrix are given by equations 4 and 5, respectively.

If the arrival time errors are uncorrelated, the numerical inversion of the covariance matrix

$$\underline{\Sigma}_{e_k} = \underline{\Delta}_k \, \underline{\Sigma}_{\varepsilon} \, \underline{\Delta}_k'$$

may be avoided, and the computation of error probabilities thus simplified. For this case, the elements of $\underline{\Sigma}_{e_k}$ are given by equation 6, namely

$$\operatorname{cov}(e_{ik}, e_{jk}) = \sigma_i^2 \, \delta_{ij} + \sigma_k^2 \, .$$

Since

$$\sum_{\substack{v \\ v \neq k}} (\sigma_i^2 \, \delta_{iv} + \sigma_k^2)(\sigma_v^{-2} \, \delta_{vj} - D\sigma_v^{-2} \, \sigma_j^{-2}) = \delta_{ij}$$

where

$$D^{-1} = \sum_{i=1}^{n} \sigma_i^{-2} ,$$

the elements of $\underline{\Sigma}_{e_k}^{-1}$ are given by

$$\sigma_i^{-2}\,\delta_{ij} - D\sigma_i^{-2}\,\sigma_j^{-2}\,. \tag{9}$$

Therefore, assuming biases have been removed from the arrival time errors, the required time difference error probabilities may be computed by means of the expression

$$P_{\underline{e}_k}(\underline{e}_k) = K_{n-1}\exp\left(-\frac{1}{2}\,\underline{e}_k'\,\underline{\Sigma}_{e_k}^{-1}\,\underline{e}_k\right) \tag{10}$$

where[1]

$$K_{n-1} = (2\pi)^{-(n-1)/2}\left|\underline{\Sigma}_{e_k}\right|^{-1/2}$$

$$= (2\pi)^{-(n-1)/2}\,D^{1/2}\prod_{i=1}^{n}\sigma_i^{-1}\,.$$

[1] Graybill, F.A., "Introduction to Matrices With Applications in Statistics," Wadsworth Publishing Co., Inc., Belmont, Ca., 1969, p. 184.

CONDITIONALLY NORMAL ERRORS

In general, where arrival time errors are non-normal, resort must be made to the transformations of equation 7 or 8 to obtain the required time difference error density. A special case of practical interest, however, may be formulated in terms of densities which are conditionally normal. It may be possible to assert that arrival time errors are normal and to specify the parameters of the density function, subject to certain conditions for which the probabilities of occurrence are known. An example would be the use of clocks of different accuracies. If the error introduced by each clock is normal, A_m, m = 1,2,..., M', denotes the event that clock m is used, and the occurrence probabilities $P(A_m)$ are known for each m, the density function for a single arrival time error ε_i would be

$$P_{\varepsilon_i}(\varepsilon_i) = \sum_{m=1}^{M'} P(\varepsilon_i | A_m)\, P(A_m) \; .$$

Clearly, this density is normal only if the use of one particular clock is certain.

For the general case we may postulate C_m, m=1,2,..., M mutually exclusive and exhaustive conditions for which $P_{\underline{\varepsilon}}(\underline{\varepsilon} | C_m)$ is jointly normal and may be denoted by $N(\underline{0}, \underline{\Sigma}_{\varepsilon|m})$. The probabilities of occurrence of each condition

$$P_m = P(C_m) \qquad m=1,2,\ldots, M$$

will be assumed known, although, in practice, estimates of these quantities would be used.

From the above discussion of the normal case

$$P_{\underline{e}_k}(\underline{e}_k | C_m) \sim N(\underline{0}, \underline{\Sigma}_{e_{k|m}})$$

where

$$\underline{\Sigma}_{e_{k|m}} = \underline{\Delta}_k \, \underline{\Sigma}_{\varepsilon|m} \, \underline{\Delta}_k'$$

and the required joint time difference error density is given by

$$p_{\underline{e}_k}(\underline{e}_k) \sim \sum_{m=1}^{M} P_m \, p_{\underline{e}_k}(\underline{e}_k | C_m)$$

$$\sim \sum_m P_m \, N(\underline{0}, \underline{\Sigma}_{e_{k|m}})$$

As in the normal case, these expressions simplify where the arrival times are uncorrelated for each of the applicable conditions. Elements of the covariance matrices $\underline{\Sigma}_{\varepsilon|m}$, $\underline{\Sigma}_{e_{k|m}}$, and $\underline{\Sigma}_{e_{k|m}}^{-1}$ are then given by

$$\sigma_{i|m}^2 \, \delta_{ij} \, ,$$

$$\sigma_{i|m}^2 \, \delta_{ij} + \sigma_{k|m}^2 \, ,$$

and $$\sigma_{i|m}^{-2} \, \delta_{ij} - D_m \, \sigma_{i|m}^{-2} \, \sigma_{j|m}^{-2} \, ,$$

respectively. Thus,

$$p_{\underline{e}_k}(\underline{e}_k) = \sum_m P_m \, K_{n-1|m} \exp\left(-\tfrac{1}{2} \, \underline{e}_k' \, \underline{\Sigma}_{e_{k|m}}^{-1} \, \underline{e}_k\right) \quad (11)$$

where

$$K_{n-1|m} = (2\pi)^{-(n-1)/2} \, D_m^{1/2} \prod_{i=1}^{n} \sigma_{i|m}^{-1} \, .$$

Care must be taken in listing the mutually exclusive conditions. If, in the above example, each of the M' clocks could be used to make any of the n time measurements, the total number of possible conditions would be $(M')!/(M'-n)!$. If there are M' types of clock and more than n clocks of each type, $(M')^n$ conditions would be possible.

The computation of time difference error probabilities may be illustrated by some examples. Consider the case of three independent time measurements using clocks of two different types, A and B. Each type of clock exhibits normal errors with zero means and variances σ_A^2 and σ_B^2 , respectively. The eight possible cases are listed below together with occurrence probabilities for a simple bimonial model.

		Measurement		
M	P_m	1	2	3
1	P_A^3	A	A	A
2	$P_A^2(1-P_A)$	A	A	B
3	$P_A^2(1-P_A)$	A	B	A
4	$P_A(1-P_A)^2$	A	B	B
5	$P_A^2(1-P_A)$	B	A	A
6	$P_A(1-P_A)^2$	B	A	B
7	$P_A(1-P_A)^2$	B	B	A
8	$(1-P_A)^3$	B	B	B

From equation 11,

$$P_{\underline{e}_k}(\underline{e}_k) = \sum_{m=1}^{8} P_m K_{2|m} \exp\left(-\frac{1}{2}\underline{e}_k' \underline{\Sigma}^{-1}_{e_{k|m}} \underline{e}_k\right)$$

which for this case may be written

$$P_{\underline{e}_k}(\underline{e}_k) = \sum_{q=0}^{1} \sum_{r=0}^{1} \sum_{s=0}^{1} P_A^{q+r+s} (1-P_A)^{3-(q+r+s)} K_{2|qrs}$$

$$\exp\left(-\frac{1}{2}\underline{e}_k' \underline{\Sigma}^{-1}_{e_{k|qrs}} \underline{e}_k\right).$$

Implementation of this formula requires only the computation of $\underline{\Sigma}^{-1}_{e_{k|qrs}}$ and $K_{2|qrs}$ for q,r,s = 1,0. Let the second observed time be chosen as reference. The cases denoted above by m=1 and m=2 are given by q=r=s=1 and q=r=1, s=0, respectively. Thus,

$$D_1 = \sigma_A^2/3 ,$$

$$\underline{\Sigma}^{-1}_{e_{2|1}} = \begin{bmatrix} 1/\sigma_A^2 - D_1/\sigma_A^4 & - D_1/\sigma_A^4 \\ - D_1/\sigma_A^4 & 1/\sigma_A^2 - D_1/\sigma_A^4 \end{bmatrix}$$

$$= 1/3\sigma_A^2 \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} ,$$

and

$$K_{2|1} = (2\sqrt{3}\,\pi\,\sigma_A^2)^{-1} .$$

$$D_2 = \frac{\sigma_A^2\,\sigma_B^2}{\sigma_A^2 + 2\sigma_B^2} ,$$

$$\Sigma_{e_{2|2}}^{-1} = \begin{bmatrix} 1/\sigma_A^2 - D_2/\sigma_A^4 & -D_2/\sigma_A^2\,\sigma_B^2 \\ -D_2/\sigma_A^2\,\sigma_B^2 & 1/\sigma_B^2 - D_2/\sigma_B^4 \end{bmatrix}$$

$$= \frac{1}{\sigma_A^2 + 2\sigma_B^2} \begin{bmatrix} 1 + \sigma_B^2/\sigma_A^2 & -1 \\ -1 & 2 \end{bmatrix} ,$$

and $K_{2|2} = (2\pi\,\sigma_A \sqrt{\sigma_A^2 + 2\sigma_B^2})^{-1}$. The remaining values of $\Sigma_{e_{2|m}}^{-1}$ and $K_{2|m}$ are obtained in the same way.

As a second example we consider the use of conditionally normal densities in the treatment of outliers. Let σ_i^2, $i=1,2,3$, be the error variances for arrival time observations given that a valid time has been observed. Given the alternative, that a "wild time" has been observed, it may be assumed that the observation is from a noninformative distribution which will be approximated by means of a normal distribution with a suitability chosen variance σ_b^2 . A listing of the appropriate variances for each of the eight possible cases is as follows.

M	P_m	Measurement 1	Measurement 2	Measurement 3
1	$P_1P_2P_3$	σ_1^2	σ_2^2	σ_3^2
2	$P_1P_2(1-P_3)$	σ_1^2	σ_2^2	σ_b^2
3	$P_1(1-P_2)P_3$	σ_1^2	σ_b^2	σ_3^2
4	$P_1(1-P_2)(1-P_3)$	σ_1^2	σ_b^2	σ_b^2
5	$(1-P_1)P_2P_3$	σ_b^2	σ_2^2	σ_3^2
6	$(1-P_1)P_2(1-P_3)$	σ_b^2	σ_2^2	σ_b^2
7	$(1-P_1)(1-P_2)P_3$	σ_b^2	σ_b^2	σ_3^2
8	$(1-P_1)(1-P_2)(1-P_3)$	σ_b^2	σ_b^2	σ_b^2

The required probabilities are given by

$$P_{\underline{e}_k}(\underline{e}_k) = \sum_{m=1}^{8} P_m \, P_{\underline{e}_k}(\underline{e}_k | C_m)$$

$$\sim \sum_{m=1}^{8} P_m \, N(\underline{0}, \underline{\Sigma}_{e_{k|m}})$$

which may be written

$$P_{\underline{e}_k}(\underline{e}_k) \sim \sum_{q=0}^{1} \sum_{r=0}^{1} \sum_{s=0}^{1} P_1^q (1-P_1)^{1-q} \, P_2^r (1-P_2)^{1-r} \, P_3^s (1-P_3)^{1-s} \, N(\underline{0}, \underline{\Sigma}_{e_{k|qrs}})$$

where the covariance matrices $\underline{\Sigma}_{\varepsilon|qrs}$ are given by

$$\underline{\Sigma}_{\varepsilon|qrs} = \begin{bmatrix} \sigma_1^2 q + \sigma_b^2(1-q) & 0 & 0 \\ 0 & \sigma_2^2 r + \sigma_b^2(1-r) & 0 \\ 0 & 0 & \sigma_3^2 s + \sigma_b^2(1-s) \end{bmatrix}$$

for q, r, s = 1, 0. The diagonal elements are therefore σ_i^2, $i = 1,2,3$, or σ_b^2, and the values of $\underline{\Sigma}^{-1}_{e_k|qrs}$ and $K_{2|qrs}$ may be computed as in the previous example. This permits the computation of $N(\underline{0}, \underline{\Sigma}_{e_k|qrs})$ for each case, and, hence, the value of $P_{\underline{e}_k}(\underline{e}_k)$ by means of the above expression.

APPENDIX

PROGRAM NOTES AND LISTING

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The number of time observations and the time chosen as reference are represented by means of the binary vector KVEC. Thus, for three observations with differences taken with respect to the second

KVEC ← 1 0 1 .

The difference operator $\underline{\Delta}_k$ is then given by the function DELOP. For example,

```
      KVEC←1 0 1
      DELOP
 1  ¯1  0
 0  ¯1  1
```

The function DEL with argument $\underline{\Sigma}_{\varepsilon}$ gives the covariance matrix

$$\underline{\Sigma}_{e_k} = \underline{\Delta}_k \underline{\Sigma}_{\varepsilon} \underline{\Delta}_k' .$$

Thus, for example,

```
      SIGEP
 0.01  0     0
 0     0.01  0
 0     0     0.01

      DEL SIGEP
 0.02  0.01
 0.01  0.02
```

The inverse of this matrix required for the numerical evaluation of the joint density function is provided by the function INDEL. Thus,

```
      INDEL SIGEP
66.66666667    ¯33.33333333
¯33.33333333    66.66666667
```

This function is therefore equivalent to the sequential application of the defined function DEL and the APL primitive function ⌹ . This latter function, however, is unavailable in many systems, and in any case, the function INDEL is more efficient.

The computation of time difference error probabilities for the normal case is by means of the function MVNPROB which incorporates all of the above functions. The left argument consists of the time difference errors expressed as a vector. The right argument may consist of either the covariance matrix for the time-of-arrival errors or a vector composed of the diagonal (nonzero) elements of the matrix.

The function CONPROB effects the multiple application of MVNPROB necessary when different conditions may prevail. The right argument of this function consists of a single matrix, each row of which is equal to the trace of the covariance matrix appropriate to one condition. The weighting of the outputs of CONPROB by the probability of occurrence of each condition is implemented by the function WTPROB. It follows that in a situation where conditional normality is applicable, the joint probability of an arbitrary vector of time differences may be computed by means of the single entry

```
PM WTPROB (X CONPROB TM)
```

where PM is a vector of occurrence probabilities, TM is the matrix fabricated row by row from the traces of the covariance matrices applicable to each condition, and X is the time difference error vector for which a probability is required.

```
      ∇DELOP[⎕]∇
     ∇ Z←DELOP;N
[1]    N←⍴KVEC
[2]    N←(⍳N)∘.=⍳N
[3]    N[;KVEC⍳0]←¯1
[4]    Z←KVEC⌿N
     ∇

      ∇DEL[⎕]∇
     ∇ Z←DEL M
[1]    Z←DELOP+.×M+.×⍉DELOP
     ∇

      ∇MVNPROB[⎕]∇
     ∇ Z←X MVNPROB Y;S;IND;K
[1]    →(1=⍴⍴Y)/CONTIN
[2]    Y←(0≠,Y)/,Y
[3]   CONTIN:S←INDEL Y
[4]    IND←+/÷Y
[5]    K←(((○2)*¯1+⍴KVEC)×IND××/Y)*¯0.5
[6]    Z←K×*¯0.5×X+.×S+.×X
     ∇

      ∇INDEL[⎕]∇
     ∇ Z←INDEL X;SA;D;SC;SD
[1]    →(1=⍴⍴X)/CONT
[2]    X←(0≠,X)/,X
[3]   CONT:SA←KVEC/÷X
[4]    SA←SA∘.×SA
[5]    D←÷+/÷X
[6]    SC←SA×D
[7]    SD←(⍳¯1+⍴X)∘.=⍳¯1+⍴X
[8]    SD←SD×SA*0.5
[9]    Z←SD-SC
     ∇

      ∇CONPROB[⎕]∇
     ∇ Z←X CONPROB TM
[1]    I←1
[2]    Z←(⍴TM)[1]⍴0
[3]   NEXT:Z[I]←X MVNPROB TM[I;]
[4]    I+←1
[5]    →(I≤⍴Z)/NEXT
     ∇

      ∇WTPROB[⎕]∇
     ∇ Z←PM WTPROB PV
[1]    Z←+/PM×PV
     ∇
```